Journal of Applied Mechanics and Technical Physics, Vol. 39, No. 5, 1998

PLANE CONFIGURATIONS IN A FLOW OF A PERFECT GAS WITH A MAXIMUM CRITICAL MACH NUMBER

L. M. Zigangareeva and O. M. Kiselev

UDC 533.6.011

The problems in the construction of bodies which, satisfying some geometric limitations, are exposed to a plane symmetric flow of a perfect (inviscid and heat-nonconducting) gas with a maximum critical Mach number M_* *are considered. Solutions are found by a numerical-analytical method with the use of the variables of the velocity hodograph. The Mach number* M_* *is found as a function of the geometric characteristics of the sought bodies on the basis of approximation of numerical data.*

The critical Mach number M_*, i.e., the minimum free-stream Mach number responsible for a flow velocity equal to the critical value, is one of the most important characteristics of bodies exposed to a gas flow. Of practical interest are bodies that satisfy some geometric restrictions and allow the maximum possible value of M_*. As bodies of this class, they do not experience the wave drag within the maximum range of free-stream velocities. We call these bodies optimal or optimal relative to M_*.

The structure of plane symmetrical optimal bodies and optimal bodies of revolution in a perfect gas flow was studied in [1, 2]. It was established that, for a wide range of geometric restrictions, the contours of optimal bodies consist of straight sections and sections on which the gas velocity equals the critical value. The shape of optimal bodies depends on the properties of a gas flow. An ideal perfect gas with the ratio of specific heats $\gamma = 1.4$ is called an air-like gas.

Various numerical and numerical-analytical methods were used in [3–6] to solve some problems of construction of plane symmetrical optimal bodies in an air-like gas flow. The results obtained were presented as numerical data and plots.

In the present work, similar problems are studied in more detail using a numerical-analytical method proposed in [7, 8], which was previously used by the authors to study optimal bodies of revolution relative to M_* [9, 10]. The formulas presented describe the dependence of M_* on the basic geometric characteristics of optimal bodies, which were obtained as a result of approximation of numerical data.

1. We consider an attached infinite steady-state potential ideal gas flow around cylindrical bodies with a plane of symmetry Ω. In the plane $z = x + iy$ perpendicular to the generatrices of the cylindrical surface of the body, the flow is assumed to be symmetric about the x axis, which belongs to Ω and is parallel to the free-stream velocity vector.

Let L_1 and H_1 be the lengths of the projections of the body onto the x and y axes, respectively, S_1 is the area of the body cut by the z plane, and θ_w is the angle of inclination of the velocity vector to the x axis on the surface of the body.

Problem A. Among symmetrical bodies which satisfy one of the conditions

$$H_1/L_1 \geqslant k_0, \qquad S_1/L_1^2 \geqslant l_0 \tag{1.1}$$

Institute of Mathematics and Mechanics at the Kazan' State University, Kazan' 420008. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 39, No. 5, pp. 106–115, September–October, 1998. Original article submitted December 3, 1996.

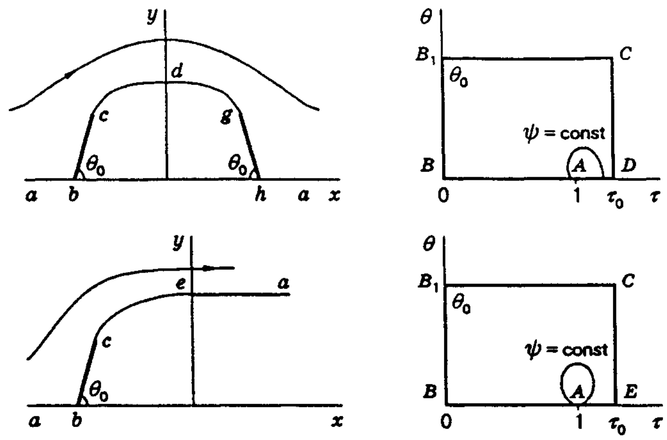

Fig. 1

and the condition

$$\max|\theta_w| \leqslant \theta_0 \leqslant \pi/2 \tag{1.2}$$

(k_0, l_0, and θ_0 are given constants, and k_0 and $l_0 < \tan\theta_0$), we have to find a body that ensures the maximum value of M_*.

Problem A_0 is obtained from the previous problem by removing condition (1.2). It was considered for the first time by Gilbarg and Shiffman [1]. According to [1], in the z plane the contour of a symmetrical body, which satisfies one of conditions (1.1) and ensures the maximum M_*, consists of two equal straight sections perpendicular to the x axis and to the streamlines that connect these sections and on which the gas velocity is equal to the critical value. In other words, problem A_0 reduces to the problem of a symmetrical flow past a flat plate in accordance with the Ryabushinskii pattern [11], with the critical velocity on the free surface. Similarly, problem A reduces to the problem of a symmetrical flow around a wedge with half-angle θ_0 in accordance with the same scheme. (The solution of problem A for $\theta_0 = \pi/2$ coincides with that of problem A_0.)

Some results of a numerical solution of problem A_0 and a problem similar to problem A can be found in [3, 4] and [5], respectively.

Problem B. Let the section of the body by the z plane be a rectangular half-band of width H_2, which is symmetric about the x axis. It is required to deform the head part (adjacent to the end face) of the body so that the resultant symmetrical body had an attached flow with the maximum possible value of M_* if condition (1.2) is satisfied together with one of the conditions

$$L_2/H_2 \leqslant m_0, \qquad S_2'/H_2^2 \leqslant n_0. \tag{1.3}$$

Here L_2 is the length of the head part (subjected to deformation) of the body, S_2' is the cross sectional area lost because of deformation, and m_0 and n_0 are some prescribed constants [$m_0 > (1/2)\cot\theta_0$ and $n_0 > (1/4)\cot\theta_0$]. Thus, we obtain $S_2' = L_2H_2 - S_2$, where S_2 is the area of the head part of the body. The inequalities (1.3) limit the loss of the "capacity" of the initially existing volume.

Note that the head part of the body can be equally readily considered to be the rear part.

Using the comparative theorem [1, 2], we can easily show that the construction of an optimal body that corresponds to the conditions of problem B reduces to the solution of the problem of a symmetric flow around a wedge with a half-angle θ_0 according to the Joukowski–Roshko pattern [11], with the critical velocity on the free surface.

Some results of the solution of the problem of contouring of the optimal, in terms of M_*, head or tail part of a plane symmetrical semi-infinite body with given values of L_2, H_2, and θ_0 can be found in [6].

2. In the plane $z = x + i\,y$, we consider a steady subsonic vortex-free flow of an ideal gas, which is

symmetric about the x axis, around a wedge with the half-angle $\theta_0 \leqslant \pi/2$ with the use of the Ryabushinskii and Joukowski–Roshko patterns. The left part of Fig. 1 shows flow regions that correspond to these schemes and are located above the x axis. Here bc and gh are the walls of the real and fictitious wedges, cdg and ce are the free surfaces, ea is the half-line parallel to the x axis, a is the infinitely distant point, and the point d lies on the y axis, which is an additional axis of symmetry for the flow according to the Ryabushinskii pattern. In accordance with the notation in Sec. 1, the distance between the points b and h is L_1, the ordinate of the point d is $H_1/2$, $S_1/2$ is the area bounded by the contour $bcdgh$ and the x axis, $S_2/2$ is the area bounded by the contour bce, the x axis, and the y axis passing through the point e, and L_2 and $H_2/2$ are the lengths of the projections of the contour bce onto the x and y axes, respectively.

Let λ be the reduced velocity, M is the Mach number, θ is the angle of inclination of the velocity to the x axis, λ_a and λ_c are the values of λ at an infinitely distant point and on the free surface, respectively ($\lambda_a \leqslant \lambda_c \leqslant 1$), M_a and M_c are the values of M for $\lambda = \lambda_a$ and $\lambda = \lambda_c$, and $\tau = \lambda/\lambda_a$ and $\tau_0 = \lambda_c/\lambda_a$. The right part of Fig. 1 shows the regions $\Sigma = \{(\tau,\theta) | 0 < \tau < \tau_0, 0 < \theta < \theta_0\}$ in the plane (τ,θ) that correspond to the top left quarter of the flow region according to the Ryabushinskii pattern and to the upper half of the flow region according to the Joukowski–Roshko pattern. The section BB_1 corresponds to the point of flow bifurcation b and the points A, C, D, and E correspond to the points a, c, d, and e, respectively.

We introduce the stream function ψ with the use of the relations $\tau\nu\cos\theta = \psi_y$ and $\tau\nu\sin\theta = -\psi_x$. Here $\nu = \rho/\rho_0$, ρ is the density of the gas, ρ_0 is the value of ρ in the frozen flow, the subscripts refer to partial derivatives. Under the above assumptions, the stream function ψ in the region Σ satisfies the Chaplygin equation

$$L\psi = (1 - \mathrm{M}^2)\psi_{\theta\theta} + \tau^2\psi_{\tau\tau} + \tau(1 + \mathrm{M}^2)\psi_\tau = 0. \tag{2.1}$$

For the flow according to the Ryabushinskii pattern, we have $\psi = 0$ on ABB_1CD and $\psi_\theta = 0$ on AD. For the flow according to the Joukowski–Roshko pattern, we have $\psi = 0$ on ABB_1CEA.

We consider the dependences $\nu(\tau)$ and $\mathrm{M}(\tau)$, determined by the gas properties, to be known differentiable functions, which are analytical in the vicinity of the point $\tau = 1$ ($\lambda = \lambda_a$). The coefficients of Eq. (2.1) can be expanded into power series with respect to $\zeta = \tau - 1$:

$$\tau^2 = 1 + 2\zeta + \zeta^2, \qquad 1 - \mathrm{M}^2 = \sum_{k=0}^{\infty} p_k\zeta^k, \qquad \tau(1 + \mathrm{M}^2) = \sum_{k=0}^{\infty} q_k\zeta^k,$$
$$p_0 = 1 - \mathrm{M}_a^2, \qquad p_k = -\frac{1}{k!}\frac{d^k\mathrm{M}^2}{d\tau^k}\bigg|_{\tau=1}, \qquad k = 1, 2, \ldots, \tag{2.2}$$
$$q_0 = 2 - p_0, \qquad q_1 = q_0 - p_1, \qquad q_k = -p_{k-1} - p_k, \qquad k = 2, 3, \ldots .$$

We represent the stream function ψ as $\psi = \psi^0 + \chi$, where ψ^0 is a function that describes the behavior of ψ in the vicinity of the singular point A. We introduce the variables σ and ω:

$$\sigma = (\theta^2 + \alpha^2\zeta^2)^{1/2}, \qquad \omega = \arctan\left(\frac{\theta}{\alpha\zeta}\right), \qquad \alpha = \sqrt{1 - \mathrm{M}_a^2}. \tag{2.3}$$

According to (2.3) and $\zeta = \alpha^{-1}\sigma\cos\omega$, we have

$$\sigma_\theta = \sin\omega, \qquad \sigma_\tau = \alpha\cos\omega, \qquad \omega_\theta = \sigma^{-1}\cos\omega, \qquad \omega_\tau = -\alpha\sigma^{-1}\sin\omega. \tag{2.4}$$

We seek the function ψ^0 in the form of an asymptotic expansion with respect to the small parameter σ. We assume that

$$\psi^0 = \psi_1 + \psi_2 + \ldots, \quad \psi_k = h_k(\sigma)f_k(\omega); \quad h_{k+1}(\sigma)/h_k(\sigma) \to 0, \quad \sigma \to 0, \quad k = 1, 2, \ldots \tag{2.5}$$

and require that the conditions

$$\psi_1 > 0, \qquad 0 < \omega < \pi \tag{2.6}$$

for both flow patterns and also the conditions

$$\psi_k = 0, \qquad \omega = 0, \pi, \qquad k = 1, 2, \dots \tag{2.7}$$

for the flow according to the Joukowski–Roshko pattern and the conditions

$$\psi_k = 0, \quad \omega = \pi; \qquad \psi_{k\theta} = 0, \quad \omega = 0, \qquad k = 1, 2, \dots \tag{2.8}$$

for the flow according to the Ryabushinskii pattern be satisfied.

We represent the main term in the expansion (2.5) in the form $\psi_1 = \sigma^{-n} f_1(\omega)$ (n = const, $n > 0$). Assuming that $\zeta = \alpha^{-1}\sigma\cos\omega$ in (2.2) and using (2.4), we can show that, for $\psi_1 = \sigma^{-n} f_1(\omega)$,

$$L\psi_1 = R_1 + \Delta R_1, \qquad R_1 = \alpha^2\sigma^{-n-2}(n^2 f_1 + f_1''), \qquad \Delta R_1 = O(\sigma^{-n-1}).$$

Equating R_1, the main term in the expansion of $L\psi_1$ with respect to powers of σ, to zero and taking into account (2.6) and (2.7) or (2.6) and (2.8), we obtain the boundary-value problem for the function $f_1(\omega)$. In particular, for the flow according to the Joukowski–Roshko pattern, the following boundary-value problem is obtained:

$$n^2 f_1 + f_1'' = 0, \quad f_1(0) = f_1(\pi) = 0, \quad f_1(\omega) > 0, \quad 0 < \omega < \pi, \quad n > 0. \tag{2.9}$$

It is easy to see that there is a unique (to within constant factor at f_1) solution of problem (2.9): $n = 1$ and $f_1 = \sin\omega$. Thus, we obtain that $\psi_1 = \sigma^{-1}\sin\omega$.

In substituting $\psi = \psi_1 = \sigma^{-1}\sin\omega$ into (2.1), the terms of the order of σ^{-3}, which result from differentiation of ψ_1, cancel each other, and the residual of the order of σ^{-2} is left. We seek the term of the expansion (2.5) that follows ψ_1 in the form $\psi_2 = f_2(\omega)$ by requiring that, after the substitution of $\psi = \psi_1 + \psi_2$ into (2.1), the terms of the order of σ^{-2} cancel each other, and the residual of the order of σ^{-1} be left. Taking into account (2.7), we obtain the following boundary-value problem for $f_2(\omega)$:

$$\begin{gathered} \alpha^2 f_2'' = \delta_1 \sin\omega\cos\omega + \delta_2 \sin^3\omega\cos\omega, \qquad \delta_1 = 6p_1\alpha^{-1} - 12\alpha + 2\alpha q_0, \\ \delta_2 = 16\alpha - 8p_1\alpha^{-1}, \qquad f_2(0) = f_2(\pi) = 0. \end{gathered} \tag{2.10}$$

There exists the unique solution of problem (2.10):

$$\psi_2 = f_2(\omega) = -\frac{1}{16}(2\delta_1 + \delta_2)\alpha^{-2}\sin 2\omega + \frac{1}{128}\delta_2\alpha^{-2}\sin 4\omega.$$

It is natural to seek the function ψ_3 in the form $\psi_3 = \sigma f_3(\omega)$ by requiring that, after the substitution of $\psi = \psi_1 + \psi_2 + \psi_3$ into (2.1), the terms of the order of σ^{-1} cancel each other, and the residual of the order of unity be left. In this case, we obtain the following boundary-value problem for f_3:

$$f_3 + f_3'' = F_3, \qquad f_3(0) = f_3(\pi) = 0 \tag{2.11}$$

(F_3 is a known function of ω). Since the function $f = \sin\omega$ satisfies the conditions $f + f'' = 0$ and $f(0) = f(\pi) = 0$, the solution of problem (2.11) is found to an accuracy of the term $q\sin\omega$, where q = const. The resultant indeterminacy shows that the behavior of ψ in the vicinity of the single point A can be refined only with allowance for the complete boundary conditions for this function.

We now pass to the flow according to the Ryabushinskii pattern. Assuming that $\psi_1 = \sigma^{-n} f_1(\omega)$, equating to zero the main term in the expansion of $L\psi_1$ with respect to powers of σ, and taking into account (2.6) and (2.8), we obtain the boundary-value problem

$$n^2 f_1 + f_1'' = 0, \qquad f_1(\pi) = f_1'(0) = 0, \qquad f_1(\omega) > 0, \qquad 0 < \omega < \pi, \qquad n > 0. \tag{2.12}$$

The solution of problem (2.12) has the form $n = 1/2$ and $f_1 = \cos(\omega/2)$. Thus, $\psi_1 = \sigma^{-1/2}\cos(\omega/2)$.

In finding the function ψ_2 there arises an indeterminacy similar to that described above for the function ψ_3 in the flow problem according to the Joukowski–Roshko pattern.

3. Knowing the function ψ^0, we can pass to the solution of the boundary-value problem for ψ, using the finite-difference method. Two approaches are possible. The first approach is based on finding the function $\chi = \psi - \psi^0$ from the solution of the corresponding boundary-value problem for the equation $L\chi = -L\psi^0$.

Using this approach, we can continue the procedure for finding the function ψ^0 that satisfies the condition $L\psi^0 \to 0$ for $\sigma \to 0$ and the boundary conditions of the corresponding problem on the sections of the boundary adjacent to the singular point A (the ambiguity of the solution is not important here). The second approach is based on the idea that the function ψ is close to ψ^0 in the vicinity of the point A. In this approach, the values of ψ^0 on a certain rectangular broken line, which separates the vicinity of the point A from the other part of the domain Σ, are used as the boundary conditions of the function ψ in solving the corresponding boundary-value problem for the equation $L\psi = 0$. Calculations demonstrated greater simplicity and reliability of the second approach, which was used to obtain the results of the present paper.

When the values of λ_a and λ_c are close to each other, the gradients of ψ near the sections AD and AE are large. This makes it necessary to transform the independent variables in a numerical solution of the problem. In particular, the following transformations [12, § 5.6], which transform Σ into $\Sigma_1 = \{(\xi,\eta)|0 < \xi < 1.\ 0 < \eta < 1\}$, are convenient:

$$\begin{gathered}\xi = F(1+\beta_1, \tau/\tau_0), \quad \eta = 1 - F(1+\beta_2, 1-\theta/\theta_0),\\ F(x,y) = \ln[(x+y)/(x-y)]\{\ln[(x+1)/(x-1)]\}^{-1}\end{gathered} \tag{3.1}$$

(the parameters β_1 and β_2 are small positive numbers).

In the domain Σ_1, we used the finite-difference scheme with a five-point approximation on a uniform rectangular grid. The method of sequential upper relaxation is used for its realization.

After determining the function $\psi(\tau,\theta)$, the transition to the physical plane is performed using the formulas

$$\nu\tau^2 z_\tau = [(\mathrm{M}^2-1)\psi_\theta + i\tau\psi_\tau]\exp(i\theta), \quad \nu\tau z_\theta = [\tau\psi_\tau + i\psi_\theta]\exp(i\theta) \tag{3.2}$$

($z = x+iy$). The derivatives ψ_τ and ψ_θ are found using the spline-approximation of the grid values of ψ.

If the function ψ is the solution of Eq. (2.1), then the values of x and y determined from (3.2) should be independent of the path of integration. This reasoning is used to rationally choose the parameters β_1 and β_2 in (3.1) and control the accuracy of calculations.

In particular, the following formula can be obtained from (3.2):

$$y = \frac{1}{\nu\tau}\left[\psi\cos\theta + \int_0^\theta (\tau\psi_\tau + \psi)\sin\theta d\theta\right] + \Omega(\tau). \tag{3.3}$$

We consider the flow according to the Joukowski–Roshko pattern. In this pattern, $\Omega(\tau) = 0$ for $0 \leqslant \tau < 1$ and $\Omega(\tau) = \text{const} > 0$ for $1 < \tau \leqslant \tau_0$. According to (3.3), for $\sigma \to 0$ ($\tau \to 1$ and $\theta \to 0$) we have

$$y \sim I = \frac{1}{\nu_a}[\psi_1 + J_1 + J_2] + \Omega(\tau), \qquad J_1 = \int_0^\theta \psi_1\theta\, d\theta, \qquad J_2 = \int_0^\theta \psi_{1\tau}\theta\, d\theta,$$

where $\psi_1 = \sigma^{-1}\sin\omega$ and ν_a is the value of ν for $\tau = 1$ ($\lambda = \lambda_a$).

Using (2.3) and (2.4), it is easy to see that

$$J_1 = \sigma[\sin\omega - g(\omega)\cos\omega], \quad J_2 = \alpha\sin\omega\cos\omega - \alpha g(\omega), \quad g(\omega) = \omega, \quad 0 \leqslant \omega < \pi/2 \quad (\zeta > 0);$$

$$g(\omega) = \omega - \pi, \qquad \pi/2 < \omega \leqslant \pi \qquad (\zeta < 0).$$

It follows that

$$I \to \frac{1}{\nu_a}\left(\sigma^{-1} + \sigma + \frac{\alpha\pi}{2}\right) \quad \text{for} \quad \zeta < 0, \quad \omega \to \pi/2,$$

$$I \to \frac{1}{\nu_a}\left(\sigma^{-1} + \sigma - \frac{\alpha\pi}{2}\right) + \Omega(\tau)\Big|_{\zeta>0} \quad \text{for} \quad \zeta > 0, \quad \omega \to \pi/2.$$

For the quantity y to be continuous on the straight line $\omega = \pi/2$ ($\zeta = 0$), it is necessary to set $\Omega(\tau) = \alpha\pi/\nu_a$ for $\zeta > 0$. Thus, for $\psi_1 = \sigma^{-1}\sin\omega$ (the function ψ_1 is determined to an accuracy of constant

TABLE 1

M_a	H_1/h	L_1/H_1	$(L_1/H_1)_1$	L_1^2/S_1	C_x
0.7	2.58882	3.90015	3.61713	4.53994	1.73478
0.8	3.93988	8.47711	7.96622	10.1306	1.45803
0.9	8.04174	27.2457	26.2955	33.1951	1.27772
0.95	16.2770	81.1159	79.7016	99.4350	1.21198
0.98	40.9940	329.112	327.669	404.376	1.17843
0.99	82.1698	937.916	938.679	1153.00	1.16810

TABLE 2

M_a	H_2/h	L_2/H_2	S_2/H_2^2	S_2'/H_2^2	C_x	QH_2h^{-1}
0.7	2.22845	1.15456	0.98182	0.17274	1.73565	1.73611
0.8	3.35659	2.61490	2.14113	0.47377	1.45822	1.45891
0.9	6.80670	8.61855	6.85351	1.76504	1.27773	1.27869
0.95	13.7467	25.8366	20.3511	5.48553	1.21198	1.21265
0.98	34.6076	105.211	82.5795	22.6313	1.17842	1.17936
0.99	69.3864	300.325	235.549	64.7756	1.16809	1.16916

factor) we have $y = \alpha\pi/\nu_a$ on AE. This relationship can be also used to control the calculation accuracy.

We introduce into consideration the cavitation number Q and the drag coefficient of the wedge C_x when the latter is exposed to a compressible fluid (gas) in accordance with one of the cavitation schemes considered: $Q = 2(p_a - p_c)/(\rho_a V_a^2)$ and $C_x = 2X/(\rho_a V_a^2 h)$. Here V_a, ρ_a, and p_a are the velocity, density, and pressure of the fluid at an infinitely distant point, p_c is the pressure in the cavity, X is the wedge drag, and h is the length of the projection of the wedge onto the y axis ($h = 2l\sin\theta_0$, where l is the length of the wedge cheek bc).

The following relationship is valid for the Joukowski–Roshko pattern [11]:

$$C_x = QH_2h^{-1}, \tag{3.4}$$

which can be also used to control the accuracy of the results obtained. We note that, for a perfect gas with the ratio of specific heats γ, for $M_c = 1$ we have

$$Q = \frac{2}{\gamma M_a^2}\left[1 - \left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}M_a^2\right)^{\gamma/(\gamma-1)}\right].$$

4. Using the above technique, a symmetrical flow of an air-like gas around a wedge was calculated according to the Ryabushinskii and Joukowski–Roshko patterns under the condition that $M_c = 1$. Systematic calculations were performed for $\theta_0 = 30$, 45, 60, 75, and 90° and $M_a = 0.7$, 0.8, 0.9, 0.95, 0.98, and 0.99 (these values of θ_0 and M_a are called the reference values). A grid obtained by division of each side of the square Σ_1 into 100 equal parts was used. The values of ψ^0 in seven internal nodes of the grid in the vicinity of the point A were used as the additional boundary values of ψ [we assume that $\psi^0 = \sigma^{-1/2}\cos(\omega/2)$ for the Ryabushinskii pattern and $\psi^0 = \sigma^{-1}\sin\omega + f_2(\omega)$ for the Joukowski–Roshko pattern]. For all the reference values of θ_0 and M_a, condition (3.4) in problem B was satisfied with an error less than 0.15%.

Some results of the solution of problems A_0 and B for $\theta_0 = \pi/2$ are listed in Tables 1 and 2, respectively (h is the length of the straight section which forms the front part of the contour of the optimal body).

By approximating the calculation data by the least-squares method, we obtained the dependences $M_* = F_1(H_1/L_1, \theta_0)$, $M_* = F_2(S_1/L_1^2, \theta_0)$, $M_* = F_3(H_2/L_2, \theta_0)$, and $M_* = F_4(H_2^2/S_2', \theta_0)$, which characterize the solutions of problems A and B for an air-like gas in the domain $\pi/6 \leqslant \theta_0 \leqslant \pi/2$, $0.7 \leqslant M_* \leqslant 1$ and have

TABLE 3

	Problem A				Problem B		
	$i=0$	$i=1$	$i=2$		$i=0$	$i=1$	$i=2$
$a_{1i}^{(1)}$	0.96658	0.00558	0.00487	$a_{1i}^{(3)}$	0.45359	0.00061	0.00952
$a_{2i}^{(1)}$	0.40108	−0.09152	−0.12140	$a_{2i}^{(3)}$	0.06727	−0.01306	−0.08314
$a_{3i}^{(1)}$	−0.40975	0.66306	0.71713	$a_{3i}^{(3)}$	−0.05215	0.08993	0.18506
$a_{1i}^{(2)}$	1.11427	0.00597	0.01947	$a_{1i}^{(4)}$	1.67544	0.03082	−0.04168
$a_{2i}^{(2)}$	0.40026	−0.10870	−0.47327	$a_{2i}^{(4)}$	−0.81137	−2.32541	0.11848
$a_{3i}^{(2)}$	−0.64369	1.37091	2.72700	$a_{3i}^{(4)}$	−0.81394	4.03693	−0.83213

the following form:

$$\begin{gathered}
F_k(t_k,\theta_0) = [1 + a_1^{(k)} t_k^{2/3} + a_2^{(k)} t_k^{4/3} + a_3^{(k)} t_k^2]^{-1}, \quad t_1 = \frac{H_1}{L_1}, \quad t_2 = \frac{S_1}{L_1^2}, \quad t_3 = \frac{H_2}{L_2}, \\
a_n^{(k)} = a_{n0}^{(k)} + a_{n1}^{(k)}(\pi/2 - \theta_0) + a_{n2}^{(k)}(\pi/2 - \theta_0)^5, \quad k = 1,\,2,\,3, \quad n = 1,\,2,\,3, \\
F_4(t_4,\theta_0) = [1 + a_1^{(4)} p + a_2^{(4)} p^2 + a_3^{(4)} p^3]^{-1}, \quad t_4 = \frac{H_2^2}{S_2'}, \quad p = 0.1\Big(\frac{1}{t_4} - \frac{1}{4}\cot\theta_0\Big)^{-2/3}, \\
a_n^{(4)} = a_{n0}^{(4)} + a_{n1}^{(4)}(\pi/2 - \theta_0) + a_{n2}^{(4)}(\pi/2 - \theta_0)^4, \qquad n = 1,\,2,\,3.
\end{gathered} \tag{4.1}$$

The values of the coefficients $a_{ni}^{(k)}$ are listed in Table 3.

For all the reference values of the parameters θ_0 and $M_a = M_*$, the error of approximation of the numerical data by formulas (4.1) does not exceed 0.05%. For $\theta_0 = \pi/2$, in deriving formulas (4.1) we used the calculation results obtained for 12 different values of M_a within the range [0.7, 0.995], including the reference values; in this case the approximation is more accurate. In particular, the relationship

$$F_1(t_1, \pi/2) = [1 + 0.96658 t_1^{2/3} + 0.40108 t_1^{4/3} - 0.40975 t_1^2]^{-1}, \tag{4.2}$$

where $t_1 = H_1/L_1$, approximates the numerical data with an error less than 0.006%.

Figure 2 shows the dependence of M_* on $t_1 = H_1/L_1$ (curves 1–3 for $\theta_0 = \pi/2$, $\pi/4$, and $\pi/6$) and on θ_0 (curves 4–6 for $t_1 = 0.2$, 0.1, and 0.05). The dependences of M_* on t_k and θ_0 for $k = 2$, 3, and 4 are similar in shape. As t_k decreases (as M_* approaches unity), the dependence of M_* on θ_0 becomes weaker ($\partial F_k(t_k,\theta_0)/\partial\theta_0 \to 0$ for $t_k \to 0$). Each of the functions $F_k(t_k,\theta_0)$ is monotonically decreasing relative to t_k and monotonically increasing relative to θ_0, at least for $\pi/6 \leqslant \theta_0 \leqslant \pi/2$ and the values of t_k that ensure satisfaction of the condition $F_k(t_k,\theta_0) \geqslant 0.7$.

In solving problem A_0 for a perfect gas with the ratio of specific heats γ, Fisher [3] used the equation

$$\psi_{\mu\mu} + \psi_{\theta\theta} + g\psi_\mu = 0, \quad \mu = \int_1^\lambda \frac{\sqrt{1 - M^2}}{\lambda}\, d\lambda, \quad g = -\frac{\gamma+1}{2}\frac{M^4}{(1 - M^2)^{3/2}}.$$

The boundary-value problem is solved for the function $\chi = \psi - \psi^0$, where $\psi^0 = \mathrm{Im}\{i[\theta + i(\mu - \mu_a)]\}^{-1/2}$ and μ_a is the value of μ for $\lambda = \lambda_a$. Calculations were performed for $\gamma = 1.4$ and three values of M_a. According to [3], we have $L_1/H_1 = 4.30$, 7.87, and 14.36 for $M_a = 0.72$, 0.80, and 0.86, respectively. The method of the present paper yields $L_1/H_1 \doteq 4.49$, 8.48, and 15.70 for the same values of M_a. This difference in the results can be explained by the use of a rather rough grid in [3] and by the need to eliminate a part of the semi-infinite region of variation of the parameters μ and θ ($-\infty < \mu \leqslant 0$).

The method used by Schwendeman et al. [4] to solve problem A_0 is close to the method of the present

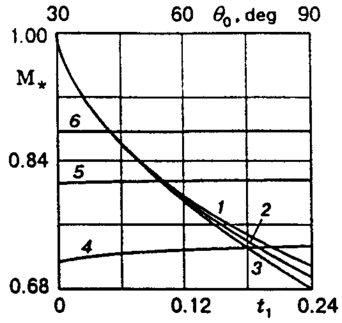

Fig. 2

paper. The results are plotted as H_1/L_1 versus M_a for $\gamma = 1.4$, which is in good agreement with our points.

For a perfect gas with the ratio of specific heats γ, the following formula is obtained in solving problem A_0 on the basis of the transonic theory of small perturbations [4]:

$$\frac{1 - M_a^2}{M_a^4} = K(H_1/L_1)^{2/3}, \qquad K = \left[\pi \frac{\gamma+1}{2} \frac{\Gamma(5/6)}{\Gamma(4/3)}\right]^{2/3}. \tag{4.3}$$

The values of L_1/H_1, calculated in accordance with (4.3) for $\gamma = 1.4$ and $K = 1.93353$, are listed in Table 1 as $(L_1/H_1)_1$. The difference between the values of L_1/H_1 and $(L_1/H_1)_1$ decreases monotonically as M_a increases from 8% for $M_a = 0.7$ to 0.08% for $M_a = 0.99$.

In accordance with (4.3), the following asymptotic relationship is valid for an exact solution of problem A_0:

$$1 - M_a^2 \sim K(H_1/L_1)^{2/3}, \qquad M_a \to 1. \tag{4.4}$$

At the same time, according to (4.2) we have

$$1 - M_a^2 \sim 2a_{10}^{(1)} t_1^{2/3} = 1.93316(H_1/L_1)^{2/3}, \qquad M_a \to 1, \qquad \gamma = 1.4.$$

Thus, the coefficient $a_{10}^{(1)}$ found as a result of approximation of numerical data, differs only by 0.02% from the value that ensures satisfaction of the asymptotic relationship (4.4) for $\gamma = 1.4$. This supports the reliability of our results.

Brutyan and Lyapunov [5] considered a problem similar to problem A. The solution of the problem of flow of an air-like gas around a wedge according to the Ryabushinskii pattern was constructed by the finite-difference method in the physical plane for prescribed parameters $\beta = l\cos\theta_0/L_1$ and M_a. The plots for M_* versus S_1/L_1^2 for $\beta = 0.2$ and θ_0 versus β for $Q = 0.5$, 0.9, and 1.5 are presented in [5] for an optimal body relative to M_*.

Mackie [13] was the first to study the problem of a symmetrical gas flow around a wedge according to the Joukowski–Roshko pattern at supersonic velocity on the free surface, which forms the basis of the solution of problem B. Using the method of separation of variables in combination with the method of contour integrals, Mackie derived analytical relations for the stream function ψ, which have the form of series whose terms contain hypergeometric functions. The flow parameters based on these relations were not calculated.

The same problem was considered by Shcherbakov [6] who constructed a function ψ_0 that describes the behavior of the stream function $\psi(\lambda, \theta)$ in the vicinity of the singular point and vanishes at the boundary of the domain of variation of the velocity hodograph. The function $\chi = \psi - \psi_0$ satisfies the equation $L\chi = -L\psi_0$ ($L\psi = 0$ is the Chaplygin equation) and homogeneous boundary conditions. To find it, we use an approach based on the construction of the Green function. As a result, χ is found as a series whose terms contain multipliers, which are hypergeometric functions, and also simple and iterated integrals with integrands which contain the same functions. The resultant representation $\psi = \psi_0 + \chi$ is used to calculate the characteristics of the flow considered. The main result of [6] is the plot of $H_2/2L_2$ versus $|\theta_0|$ ($0 \leqslant |\theta_0| \leqslant 90°$) for $M_* =$

0.3, 0.4, ..., 0.9 for an air-like gas.

The method of the present paper yields results which are in reasonable agreement with the plots presented in [5, 6].

5. Thus, the basic result of the present paper is analytical functions of the critical Mach number M_* versus t_k and θ_0, which were found for the first time and characterize the solution of problems A and B for an air-like gas in the range of arguments which is most important for practice. We note that the function $F_1(t_1, \theta_0)$ for $\theta_0 < \pi/2$ and also the functions $F_2(t_2, \theta_0)$ and $F_4(t_4, \theta_0)$ have not yet been studied.

Obviously, for arbitrary plane symmetrical bodies of finite length, which satisfy condition (1.2) and are exposed to an attached symmetrical potential flow of an air-like gas, the following relations are valid:

$$M_* \leqslant F_1(H_1/L_1, \theta_0), \qquad M_* \leqslant F_2(S_1/L_1^2, \theta_0). \tag{5.1}$$

Similarly, for arbitrary plane symmetrical semi-infinite bodies obtained by means of deformation of the front part of a rectangular band, which satisfy condition (1.2) and are exposed to an attached potential flow of an air-like gas, the following relations are valid:

$$M_* \leqslant F_3(H_2/L_2, \theta_0), \qquad M_* \leqslant F_4(H_2^2/S_2', \theta_0). \tag{5.2}$$

A strict equality in (5.1) and (5.2) is fulfilled only for optimal bodies relative to M_*.

The authors are thankful to G. Yu. Stepanov for his attention to this work.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 96-01-00123).

REFERENCES

1. D. Gilbarg and M. Shiffman, "On bodies achieving extreme values of the critical Mach number. I," *J. Ration. Mech. Anal.*, **3**, No. 2, 209–230 (1954).
2. A. N. Kraiko, "Plane and axisymmetrical configurations in a flow with the maximum critical Mach number," *Prikl. Mat. Mekh.*, **51**, No. 6, 941–950 (1987).
3. D. D. Fisher, "Calculation of subsonic cavities with sonic free streamlines," *J. Math. Phys.*, **42**, No. 1, 14–26 (1963).
4. D. W. Schwendeman, M. C. A. Kropinski, and J. D. Cole, "On the construction and calculation of optimal nonlifting critical airfoils," *ZAMP*, **44**, 556–571 (1993).
5. M. A. Brutyan and S. V. Lyapunov, "Optimization of the shape of symmetrical plane bodies with a view for increasing the critical Mach number," *Uch. Zap. TsAGI*, **12**, No. 5, 10–22 (1981).
6. S. A. Shcherbakov, "Calculation of the fore or aft part of a plane body in a subsonic flow with the maximum possible critical Mach number," *Uch. Zap. TsAGI*, **19**, No. 4, 10–18 (1988).
7. O. M. Kiselev, "Calculation of jet flows of gas with the use of the variables of the velocity hodograph," in: *Abstracts of IV All-Union School "High-Velocity Hydrodynamics"* [in Russian], Izd. Chuvash. Univ., Cheboksary (1989), p. 34.
8. L. M. Zigangareeva and O. M. Kiselev, "Calculation of a subsonic compressible cavitation flow around a circular cone," *Prikl. Mat. Mekh.*, **58**, No. 4, 93–107 (1994).
9. L. M. Zigangareeva and O. M. Kiselev, "Separated subsonic ideal gas flow around a body with the maximum critical Mach numbers," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 3, 166–172 (1996).
10. L. M. Zigangareeva and O. M. Kiselev, "Semi-infinite bodies of revolution in a flow with the maximum critical Mach number," *Prikl. Mat. Mekh.*, **61**, No. 1, 97–107 (1997).
11. M. I. Gurevich, *Theory of Jets of Perfect Fluid* [in Russian], Fizmatgiz, Moscow (1961).
12. D. Anderson, J. Tannehill, and R. Pletcher, *Computational Fluid Mechanics and Heat Transfer*, Hemisphere, New York (1984).
13. A. G. Mackie, "The solution of boundary value problems for a general hodograph equation," *Proc. Cambridge Philos. Soc.*, **54**, Part 4, 538–553 (1958).